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The sh Lie structure of Poisson brackets in field theory

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Abstract

A general construction of an sh Lie algebra (L_∞-algebra) from a homological resolution of a Lie algebra is given. It is applied to the space of local functionals equipped with a Poisson bracket, induced by a bracket for local functions along the lines suggested by Gel'fand, Dickey and Dorfman. In this way, higher order maps are constructed which combine to form an sh Lie algebra on the graded differential algebra of horizontal forms. The same construction applies for graded brackets in field theory such as the Batalin-Fradkin-Vilkovisky bracket of the Hamiltonian BRST theory or the Batalin-Vilkovisky antibracket.

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1 Introduction

In field theories, an important class of physically interesting quantities, such as the action or the Hamiltonian, are described by local functionals, which are the integral over some region of spacetime (or just of space) of local functions, i.e., functions which depend on the fields and a finite number of their derivatives. It is often more convenient to work with these integrands instead of the functionals, because they live on finite dimensional spaces. The price to pay is that one has to consider equivalence classes of such integrands modulo total divergences in order to have a one-to-one correspondence with the local functionals.

The approach to Poisson brackets in this context, pioneered by Gel'fand, Dickey and Dorfman, is to consider the Poisson brackets for local functionals as being induced by brackets for local functions, which are not necessarily strictly Poisson. We will analyze here in detail the structure of the brackets for local functions corresponding to the Poisson brackets for local functionals. More precisely, we will show that these brackets will imply higher order brackets combining into a strong homotopy Lie algebra.

The paper is organized as follows:

In the case of a homological resolution of a Lie algebra, it is shown that a skew-symmetric bilinear map on the resolution space inducing the Lie bracket of the algebra extends to higher order 'multi-brackets' on the resolution space which combine to form an L_∞-algebra (strong homotopy Lie algebra or sh Lie algebra). For completeness, the definition of these algebras is briefly recalled. For a highly connected complex which is not a resolution, the same procedure yields part of an sh Lie algebra on the complex with corresponding multi-brackets on the homology .

This general construction is then applied in the following case.

If the horizontal complex of the variational bicomplex is used as a resolution for local functionals equipped with a Poisson bracket, we can construct an sh Lie algebra (of order the dimension of the base space plus two) on the graded differential algebra of horizontal forms. The construction applies not only for brackets in Darboux coordinates as well as for non-canonical brackets such as those of the KDV equation, but also in the presence of Grassmann odd fields for graded even brackets, such as the extended Poisson bracket appearing in the Hamiltonian formulation of the BRST theory, and for graded odd brackets, such as the antibracket of the Batalin-Vilkovisky formalism.

2 Sh Lie algebras from homological resolution of Lie algebras

2.1 Construction

Let $\mathcal{F}$ be a vector space and (X_*, l_1) a homological resolution thereof, i.e., X_* is a graded vector space, l_1 is a differential and lowers the grading by one with $\mathcal{F} \simeq H_0(l_1)$ and $H_k(l_1) = 0$ for $k > 0$. The complex (X_*, l_1) is called the resolution space. (We are *not* using the term 'resolution' in a categorical sense.)

Let $\mathcal{C}_*$ and $\mathcal{B}_*$ denote the l_1 cycles (respectively, boundaries) of X_*. Recall that by convention X_0 consists entirely of cycles, equivalently $X_{-1} = 0$. Hence, we have a decomposition

$$X_0 = \mathcal{B}_0 \oplus \mathcal{K}, \tag{1}$$

with $\mathcal{K} \simeq \mathcal{F}$.

We may rephrase the above situation in terms of the existence of a contracting homotopy on (X_*, l_1) specifying a homotopy inverse for the canonical homomorphism $\eta : X_0 \longrightarrow H_0(X_*) \simeq \mathcal{F}$. We may regard $\mathcal{F}$ as a differential graded vector space $\mathcal{F}_*$ with $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_k = 0$ for $k > 0$; the differential is given by the trivial map. We then consider the chain map $\eta : X_* \longrightarrow \mathcal{F}_*$ with homotopy inverse $\lambda : \mathcal{F}_* \longrightarrow X_*$; i.e., we have that $\eta \circ \lambda = 1_{\mathcal{F}_*}$ and that $\lambda \circ \eta \sim 1_{X_*}$ via a chain homotopy $s : X_* \longrightarrow X_*$ with $\lambda \circ \eta - 1_{X_*} = l_1 \circ s + s \circ l_1$. Observe that this equation takes on the form $\lambda \circ \eta - 1_{X_*} = l_1 \circ s$ on X_0.

We may summarize all of the above with the commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & X_2 & \overset{s}{\underset{l_1}{\leftrightarrows}} & X_1 & \overset{s}{\underset{l_1}{\leftrightarrows}} & X_0 \\
& \lambda \uparrow\downarrow \eta & & \lambda \uparrow\downarrow \eta & & \lambda \uparrow\downarrow \eta \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H_0 = \mathcal{F}.
\end{array}$$

It is clear that $\eta(b) = 0$ for $b \in \mathcal{B}_0$.

Let $(-1)^\sigma$ be the signature of a permutation σ and $unsh(k, p)$ the set of

permutations σ satisfying

$$\underbrace{\sigma(1) < \ldots < \sigma(k)}_{\text{first } \sigma \text{ hand}} \qquad \text{and} \qquad \underbrace{\sigma(k+1) < \ldots < \sigma(k+p).}_{\text{second } \sigma \text{ hand}}$$

We will be concerned with skew-symmetric linear maps

$$\tilde{l}_2 : X_0 \otimes X_0 \longrightarrow X_0 \tag{2}$$

that satisfy the properties

$$(i)\ \tilde{l}_2(c, b_1) = b_2 \tag{3}$$
$$(ii) \sum_{\sigma \in unsh(2,1)} (-1)^\sigma \tilde{l}_2(\tilde{l}_2(c_{\sigma(1)}, c_{\sigma(2)}), c_{\sigma(3)}) = b_3 \tag{4}$$

where $c, c_i \in X_0$, $b_i \in \mathcal{B}_0$ and with the additional structures on X_* as well as on $\mathcal{F}$ that such maps will yield.

We begin with

Lemma 1 *The existence of a skew-symmetric linear map $\tilde{l}_2 : X_0 \bigotimes X_0 \longrightarrow X_0$ that satisfies condition (i) is equivalent to the existence of a skew-symmetric linear map $[\cdot,\cdot] : \mathcal{F} \bigotimes \mathcal{F} \longrightarrow \mathcal{F}$.*

Proof: $\tilde{l}_2$ induces a linear mapping on $\mathcal{F} \bigotimes \mathcal{F}$ via the diagram

$$\begin{array}{ccc} X_0 \bigotimes X_0 & \xrightarrow{\tilde{l}_2} & X_0 \\ \lambda \otimes \lambda \uparrow & & \downarrow \eta \\ \mathcal{F} \bigotimes \mathcal{F} & \xrightarrow{[\cdot,\cdot]} & \mathcal{F}. \end{array}$$

The fact that $\tilde{l}_2$ satisfies condition (i) guarantees that $[\cdot,\cdot]$ is well-defined on the homology classes.

Conversely, given $[\cdot,\cdot] : \mathcal{F} \bigotimes \mathcal{F} \longrightarrow \mathcal{F}$, define $\tilde{l}_2 = \lambda \circ [\cdot,\cdot] \circ \eta \otimes \eta$. It is clear that $\tilde{l}_2$ is skew-symmetric because $[\cdot,\cdot]$ is, and condition (i) is satisfied in the strong sense that $\tilde{l}_2(c, b) = 0$. $\square$

Lemma 2 *Assume that $\tilde{l}_2 : X_0 \bigotimes X_0 \longrightarrow X_0$ satisfies condition (i). Then the induced bracket on $\mathcal{F}$ is a Lie bracket if and only if $\tilde{l}_2$ satisfies condition (ii).*

Proof: Assume that the induced bracket on $\mathcal{F}$ is a Lie bracket; recall that the bracket is given by the composition $\eta \circ \tilde{l}_2 \circ \lambda \otimes \lambda$. The Jacobi identity takes on the form, for arbitrary $f_i \in \mathcal{F}$,

$$\sum_{\sigma \in unsh(2,1)} (-1)^\sigma (\eta \circ \tilde{l}_2 \circ \lambda \otimes \lambda)(\eta \circ (\tilde{l}_2 \otimes 1) \circ (\lambda \otimes \lambda \otimes 1)$$

$$(f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes f_{\sigma(3)}) = 0$$

$$\Leftrightarrow \sum_{\sigma \in unsh(2,1)} (-1)^\sigma (\eta \circ \tilde{l}_2 \circ \lambda \otimes \lambda)(\eta \circ \tilde{l}_2(\lambda(f_{\sigma(1)}) \otimes \lambda(f_{\sigma(2)})) \otimes f_{\sigma(3)}) = 0$$

$$\Leftrightarrow \sum_{\sigma \in unsh(2,1)} (-1)^\sigma \eta \circ \tilde{l}_2(\lambda \circ \eta \circ \tilde{l}_2(\lambda(f_{\sigma(1)}) \otimes \lambda(f_{\sigma(2)})) \otimes \lambda(f_{\sigma(3)})) = 0$$

$$\Leftrightarrow \sum_{\sigma \in unsh(2,1)} (-1)^\sigma \eta \circ \tilde{l}_2((1 + l_1 \circ s) \circ \tilde{l}_2(\lambda(f_{\sigma(1)}) \otimes \lambda(f_{\sigma(2)})) \otimes \lambda(f_{\sigma(3)})) = 0$$

$$\Leftrightarrow \sum_{\sigma \in unsh(2,1)} (-1)^\sigma \eta \circ \tilde{l}_2(\tilde{l}_2(\lambda(f_{\sigma(1)}) \otimes \lambda(f_{\sigma(2)})) \otimes \lambda(f_{\sigma(3)}))$$

$$+ \sum_{\sigma \in unsh(2,1)} (-1)^\sigma \eta \circ \tilde{l}_2(l_1 \circ s \circ \tilde{l}_2(\lambda(f_{\sigma(1)}) \otimes \lambda(f_{\sigma(2)})) \otimes \lambda(f_{\sigma(3)})) = 0$$

$$\Leftrightarrow \eta(\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \tilde{l}_2(\tilde{l}_2(\lambda(f_{\sigma(1)}) \otimes \lambda(f_{\sigma(2)})) \otimes \lambda(f_{\sigma(3)}))) + \eta(b) = 0$$

But $\eta(b) = 0$ and so

$$\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \tilde{l}_2(\tilde{l}_2(\lambda(f_{\sigma(1)}) \otimes \lambda(f_{\sigma(2)})) \otimes \lambda(f_{\sigma(3)})) = b' \in \mathcal{B}_0. \tag{5}$$

The converse follows from a similar calculation. □

Remark : The interesting case here is when $\mathcal{F}$ is only known as $X_0/\mathcal{B}_0$ and the only characterization of the Lie bracket $[\cdot,\cdot]$ in $\mathcal{F}$ is as the bracket induced by $\tilde{l}_2$. An important particular case, to be considered elsewhere, occurs when X_0 is a Lie algebra $\mathcal{G}$ with Lie bracket L_2 and $\mathcal{B}_0$ a Lie ideal. The bracket $\tilde{l}_2$ is defined by choosing a vector space complement $\mathcal{K}$ of the ideal $\mathcal{B}_0$ in X_0 and then projecting the Lie bracket L_2 onto $\mathcal{K}$. Hence, $L_2(c_1, c_2) = \tilde{l}_2(c_1, c_2) + b(c_1, c_2)$, where $b(c_1, c_2)$ is a well-defined element of $\mathcal{B}_0$. Indeed, by definition, property (i) holds with zero on the right hand side. Property (ii) follows from the Jacobi identity for L_2 :

$$0 = \sum_{\sigma \in unsh(2,1)} (-1)^\sigma L_2(L_2(c_{\sigma(1)}, c_{\sigma(2)}), c_{\sigma(3)})$$

$$= \sum_{\sigma \in unsh(2,1)} (-1)^\sigma [\tilde{l}_2(L_2(c_{\sigma(1)}, c_{\sigma(2)}), c_{\sigma(3)}) + b(L_2(c_{\sigma(1)}, c_{\sigma(2)}), c_{\sigma(3)})]$$

$$= \sum_{\sigma \in unsh(2,1)} (-1)^\sigma [\tilde{l}_2(\tilde{l}_2(c_{\sigma(1)}, c_{\sigma(2)}), c_{\sigma(3)}) + b(L_2(c_{\sigma(1)}, c_{\sigma(2)}), c_{\sigma(3)})]. \qquad (6)$$

We now turn our attention to the maps that $\tilde{l}_2$ induces on the complex X_*.

Lemma 3 *A skew-symmetric linear map* $\tilde{l}_2 : X_0 \bigotimes X_0 \longrightarrow X_0$ *that satisfies condition* (i) *extends to a degree zero skew-symmetric chain map* $l_2 : X_* \bigotimes X_* \longrightarrow X_*$.

Proof: We first extend $\tilde{l}_2$ to a linear map $l_2 : X_1 \bigotimes X_0 \longrightarrow X_1$ by the following: let $x_1 \otimes x_0 \in X_1 \bigotimes X_0$. Then $l_1(x_1 \otimes x_0) = l_1 x_1 \otimes x_0 + x_1 \otimes l_1 x_0 = l_1 x_1 \otimes x_0 \in X_0 \bigotimes X_0$. So we have that $l_2 l_1(x_1 \otimes x_0) = \tilde{l}_2(l_1 x_1 \otimes x_0) = b$ by condition (i). Write $b = l_1 z_1$ for $z_1 \in X_1$ and define $l_2(x_1 \otimes x_0) = z_1$. Also extend l_2 to $X_0 \bigotimes X_1$ by skew-symmetry: $l_2(x_0 \otimes x_1) = -l_2(x_1 \otimes x_0)$. Note that l_2 is a chain map by construction.

Now assume that l_2 is defined and is a chain map on elements of degree less than or equal to n in $X_* \bigotimes X_*$. Let $x_p \otimes x_q \in X_p \bigotimes X_q$ where $p+q = n+1$. Because $l_1(x_p \otimes x_q)$ has degree n, $l_2 l_1(x_p \otimes x_q)$ is defined. We have that

$$\begin{aligned} l_1 l_2 l_1(x_p \otimes x_q) &= \\ &= l_1 l_2 [l_1 x_p \otimes x_q + (-1)^p x_p \otimes l_1 x_q] \\ &= l_2 l_1 [l_1 x_p \otimes x_q + (-1)^p x_p \otimes l_1 x_q] \quad \text{because } l_2 \text{ is a chain map} \\ &= l_2 [l_1 l_1 x_p \otimes x_q + (-1)^{p-1} l_1 x_p \otimes l_1 x_q \\ &\quad + (-1)^p l_1 x_p \otimes l_1 x_q + x_p \otimes l_1 l_1 x_q] = 0 \end{aligned} \qquad (7)$$

because $l_1^2 = 0$ and $(-1)^p$ and $(-1)^{p-1}$ have opposite parity. Thus $l_2 l_1(x_p \otimes x_q)$ is a cycle in X_n and so there is an element $z_{n+1} \in X_{n+1}$ with $l_1 z_{n+1} = l_2 l_1(x_p \otimes x_q)$. Define $l_2(x_p \otimes x_q) = z_{n+1}$. As before, extend l_2 to $X_q \bigotimes X_p$ by skew-symmetry and note that l_2 is a chain map by construction. □

Remark : We will be concerned with (graded) skew-symmetric maps $f_n : \bigotimes^n X_* \longrightarrow X_*$ that have been extended to maps $f_n : \bigotimes^{n+k} X_* \longrightarrow \bigotimes^k X_*$ via the equation

$$\begin{aligned} &f_n(x_1 \otimes \ldots \otimes x_{n+k}) = \\ &\sum_{unsh(n,k)} (-1)^\sigma e(\sigma) f_n(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}) \otimes x_{\sigma(n+1)} \otimes \ldots \otimes x_{\sigma(n+k)}, \end{aligned} \qquad (8)$$

where $e(\sigma)$ is the Koszul sign (see e.g. [16]). This extension arises from the skew-symmetrization of the extension of a linear map as a skew coderivation on the tensor coalgebra on the graded vector space X_* [15]. The extension of the differential l_1 assumed in the previous lemma is equivalent to the one given by this construction. We assume for the remainder of this section that all maps have been extended in this fashion when necessary; moreover, we will use the uniqueness of such extensions when needed.

When the original skew-symmetric map $\tilde{l}_2$ satisfies both conditions (i) and (ii), there is a very rich algebraic structure on the complex X_*.

Proposition 4 *A skew-symmetric linear map $\tilde{l}_2 : X_0 \bigotimes X_0 \longrightarrow X_0$ that satisfies conditions (i) and (ii) extends to a chain map $l_2 : X_* \bigotimes X_* \longrightarrow X_*$; moreover, there exists a degree one map $l_3 : X_* \bigotimes X_* \bigotimes X_* \longrightarrow X_*$ with the property that $l_1 l_3 + l_3 l_1 + l_2 l_2 = 0$.*

Here, we have suppressed the notation that is used to indicate the indexing of the summands over the appropriate unshuffles as well as the corresponding signs. They are given explicitly in Definition 5 below.

Proof: We extend $\tilde{l}_2$ to $l_2 : X_* \bigotimes X_* \longrightarrow X_*$ as in the previous lemma. In degree zero, $l_2 l_2(x_1 \otimes x_2 \otimes x_3)$ is equal to a boundary b in X_0 by condition (ii). There exists an element $z \in X_1$ with $l_1 z = b$ and so we define $l_3(x_1 \otimes x_2 \otimes x_3) = -z$. Because $l_1 = 0$ on $X_0 \bigotimes X_0 \bigotimes X_0$, we have that $l_1 l_3 + l_2 l_2 + l_3 l_1 = 0$.

Now assume that l_3 is defined up to degree p in $X_* \bigotimes X_* \bigotimes X_*$ and satisfies the relation $l_1 l_3 + l_2 l_2 + l_3 l_1 = 0$. Consider the map $l_2 l_2 + l_3 l_1$ which is inductively defined on elements of degree $p+1$ in $X_* \bigotimes X_* \bigotimes X_*$. We have that $l_1[l_2 l_2 + l_3 l_1] = l_1 l_2 l_2 + l_1 l_3 l_1 = l_2 l_1 l_2 + l_1 l_3 l_1 = l_2 l_2 l_1 + l_1 l_3 l_1 = [l_2 l_2 + l_1 l_3] l_1 = -l_3 l_1 l_1 = 0$. Thus the image of $l_2 l_2 + l_3 l_1$ is a boundary in X_p which is then the image of an element, say $z_{p+1} \in X_{p+1}$. Define now l_3 applied to the original element of degree $p+1$ in $X_* \bigotimes X_* \bigotimes X_*$ to be this element z_{p+1}. $\square$

In the proof of the proposition above, we made repeated use of the relation $l_1 l_2 - l_2 l_1 = 0$ when extended to an arbitrary number of variables. We may justify this by observing that the map $l_1 l_2 - l_2 l_1$ is the commutator of the skew coderivations l_1 and l_2 and is thus a coderivation; it follows that the extension of this map must equal the extension of the 0 map.

The relations among the maps l_i that were generated in the previous results are the first relations that one encounters in an sh Lie algebra (L_∞ algebra). Let us recall the definition [17, 16].

Definition 5 *An sh Lie structure on a graded vector space X_* is a collection of linear, skew symmetric maps $l_k : \bigotimes^k X_* \longrightarrow X_*$ of degree $k-2$ that satisfy the relation*

$$\sum_{i+j=n+1} \sum_{unsh(i,n-i)} e(\sigma)(-1)^{\sigma}(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), \ldots, x_{\sigma(n)}) = 0, \tag{9}$$

where $1 \leq i, j$.

Remark: Recall that the suspension of a graded vector space X_*, denoted by $\uparrow X_*$, is the graded vector space defined by $(\uparrow X_*)_n = X_{n-1}$ while the desuspension of X_* is given by $(\downarrow X_*)_n = X_{n+1}$. It can be shown, [17, 16], Theorem 2.3, that the data in the definition is equivalent to

a) the existence of a degree -1 coderivation D on $\bigwedge^* \uparrow X$, the cocommutative coalgebra on the graded vector space $\uparrow X$, with $D^2 = 0$.

and to

b) the existence of a degree $+1$ derivation δ on $\bigwedge^* \uparrow X^*$, the exterior algebra on the suspension of the degree-wise dual of X_*, with $\delta^2 = 0$. In this case, we require that X_* be of finite type.

Let us examine the relations in the above definition independently of the underlying vector space X_* and write them in the form

$$\sum_{i+j=n+1} (-1)^{i(j-1)} l_i l_j = 0$$

where we are assuming that the sums over the appropriate unshuffles with the corresponding signs are incorporated into the definition of the extended maps l_k. We will require the fact that the map

$$\sum_{i,j>1} (-1)^{i(j-1)} l_j l_i : \bigotimes^n X_* \longrightarrow X_*$$

is a chain map in the following sense:

Lemma 6 *Let $\{l_k\}$ be an sh Lie structure on the graded vector space X_*. Then*

$$l_1 \sum_{i,j>1} (-1)^{(j-1)i} l_j l_i = (-1)^{n-1} (\sum_{i,j>1} (-1)^{(j-1)i} l_j l_i) l_1 \tag{10}$$

where $i + j = n + 1$.

Proof: Let us reindex the left hand side of the above equation and write it as

$$l_1 \sum_{i=2}^{n-1} (-1)^{(n-i)i} l_{n-i+1} l_i$$

which, after applying the sh Lie relation to the composition $l_1 l_{n-i+1}$, is equal to

$$\sum_{i=2}^{n-1} \sum_{k=2}^{n-i} (-1)^{(n-i)i} (-1)^{(k-1)(n-i-k)+1} l_k l_{n-i-k+2} l_i.$$

On the other hand, the right hand side may be written as

$$(-1)^{n-1} \sum_{i=2}^{n-1} (-1)^{(i-1)(n-i+1)} l_i l_{n-i+1} l_1$$

which in turn is equal to

$$(-1)^{n-1} \sum_{i=2}^{n-1} \sum_{k=2}^{n-i} (-1)^{(i-1)(n-i+1)} (-1)^{(n-i-k+1)k} (-1)^{n-i+1} l_i l_{n-i+2-k} l_k.$$

It is clear that the two resulting expressions have identical summands and a straightforward calculation yields that the signs as well are identical. □

The argument in the previous proposition may be extended to construct higher order maps l_k so that we have

Theorem 7 *A skew-symmetric linear map $\tilde{l}_2 : X_0 \bigotimes X_0 \longrightarrow X_0$ that satisfies conditions (i) and (ii) extends to an sh Lie structure on the resolution space X_*.*

Proof: We already have the required maps l_1, l_2 and l_3 from our previous work. We use induction to assume that we have the maps l_k for $1 \leq k < n$ that satisfy the relation in the definition of an sh Lie structure. To construct the map l_n, we begin with the map

$$\sum_{i+j=n+1} (-1)^{(j-1)i} l_j l_i : \bigotimes^n X_0 \longrightarrow X_{n-3}$$

with $i, j > 1$. Apply the differential l_1 to this map to get

$$l_1 \sum_{i+j=n+1} (-1)^{(j-1)i} l_j l_i = (-1)^{n-1} \sum_{i+j=n+1} (-1)^{(j-1)i} l_j l_i l_1 = 0 \qquad (11)$$

where the first equality follows from the lemma and the second equality from the fact that l_1 is 0 on $\bigotimes^n X_0$. The acyclicity of the complex X_* will then yield, with care to preserve the desired symmetry, the map l_n on $\bigotimes^n X_*$ and it will satisfy the sh Lie relations by construction.

Finally, assume that all of the maps l_k for $k < n$ have been constructed so as to satisfy the sh Lie relations and that l_n has been constructed in a similar fashion through degree p in $\bigotimes^n X_*$. We have the map

$$\sum_{i+j=n+1} (-1)^{(j-1)i} l_j l_i : (\bigotimes^n X_*)_p \longrightarrow X_{p-3}$$

to which we may apply the differential l_1. This results in

$$\sum_{i+j=n+1} (-1)^{(j-1)i} l_1 l_j l_i = \sum_{i,j>1} (-1)^{(j-1)i} l_1 l_j l_i + (-1)^{n-1} l_1 l_n l_1$$

$$= \sum_{i,j>1} (-1)^{(j-1)i} (-1)^{n-1} l_j l_i l_1 + (-1)^{n-1} l_1 l_n l_1$$

$$= \sum_{i,j>1} (-1)^{(j-1)i} (-1)^{n-1} l_j l_i l_1 + (-1)^{n-1} (- \sum_{i,j>1} (-1)^{(j-1)i} l_j l_i l_1) = 0$$

and so again, we have the existence of l_n together with the appropriate sh Lie relations. □

Remarks : (i) It may be the case in practice that the complex X_* is truncated at height n, i.e. we have that X_* is not a resolution but rather may have non-trivial homology in degree n as well as in degree 0. In such a case, our construction of the maps l_k may be terminated by degree n obstructions. More precisely, we have that the vanishing of $H_k X_*$ for k different from 0 and n will then only guarantee the existence of the requisite maps $l_k : (\bigotimes^k X_*)_p \longrightarrow X_{p+k-2}$ for $p + k - 2 \leq n$.

(ii) If property (i) holds with zero on the right hand side, i.e. so that l_2 vanishes if one of the x_i's is in $\mathcal{B}_0$, then l_2 can be extended trivially (to be a chain map) as zero on $(\bigotimes^2 X_*)_p$ for $p > 0$. It is easy to see that in the recursive construction, one can choose similarly trivial extensions of the maps l_k for $k > 2$ to all of the resolution complex, i.e. they are defined to vanish whenever one of their arguments belongs to $\mathcal{B}_0$ or X_p for $p > 0$. Hence they induce well-defined maps $\hat{l}_k$ on $\bigotimes^k \mathcal{F}$. With these choices, each

of the defining equations of the sh Lie algebra on X_* involves only two terms, namely $l_1 l_k + l_{k-1} l_2 = 0$ for $k \geq 3$.

(iii) If $H_k X_* = 0$ for $0 < k < n$ and property (i) holds with zero on the right hand side, we have defined a map on $\bigotimes^{n+2} \mathcal{F}$ which may be non-zero. If so, the only non-trivial defining equation of the induced sh Lie algebra on $\mathcal{F}$ reduces to $\hat{l}_{n+2}\hat{l}_2 = 0$. For example, if $n = 1$, $\Sigma \hat{l}_3(\hat{l}_2(x_i, x_j), x_k, x_m) = 0$ where the sum is over all permutations of (1234) such that $i < j$ and $k < m$.

In section 3, we apply this construction in the context of Poisson brackets in field theory.

2.2 Generalization to the graded case

The above construction of an L_∞-structure on the resolution of a Lie algebra can be extended in a straightforward way to the graded case, i.e. when the Lie algebra is graded (either by $\mathbf{Z}$ or by $\mathbf{Z/2}$, the super case) and the bracket is of a fixed degree, even or odd, satisfying the appropriate graded version of skew-symmetry and the Jacobi identity. We will refer to all of these possibilities as graded Lie algebras although the older mathematical literature uses that term only for the case of a degree 0 bracket. In these situations, the resolution X_* is bigraded and the inductive steps proceed with respect to the resolution degree (see [20, 15] for carefully worked out examples).

The graded case occurs in the Batalin-Fradkin-Vilkovisky approach to constrained Hamiltonian field theories [8, 2, 7, 14] where their bracket is of degree 0 and in the Batalin-Vilkovisky anti-field formalism for mechanical systems or field theories [3, 4, 14] with their anti-bracket of degree 1.

In all these cases, one need only take care of the signs by the usual rule: when interchanging two things (operators, fields, ghosts, etc.), be sure to include the sign of the interchange.

3 Local field theory with a Poisson bracket

We first review the result that the cohomological resolution of local functionals is provided by the horizontal complex. Then, we give the definition of a Poisson bracket for local functionals. The existence of such a Poisson bracket will assure us that the conditions of the previous section hold. Hence, we

show that to the Poisson bracket for local functionals corresponds an sh Lie algebra on the graded differential algebra of horizontal forms.

3.1 The horizontal complex as a resolution for local functionals.

In this subsection, we introduce some basic elements from jet-bundles and the variational bicomplex relevant for our purpose. More details and references to the original literature can be found in [18, 19, 5, 1]. For the most part, we will follow the definitions and the notations of [18]. Although much of what we do is valid for general vector bundles, we will not be concerned with global properties. We will use local coordinates most of the time, though we will set things up initially in the global setting.

Let M be an n-dimensional manifold and $\pi : E \to M$ a vector bundle of fiber dimension k over M. Let $J^\infty E$ denote the infinite jet bundle of E over M with $\pi^\infty_E : J^\infty E \to E$ and $\pi^\infty_M : J^\infty E \to M$ the canonical projections. The vector space of smooth sections of E with compact support will be denoted ΓE. For each (local) section ϕ of E, let $j^\infty\phi$ denote the induced (local) section of the infinite jet bundle $J^\infty E$.

The restriction of the infinite jet bundle over an appropriate open $U \subset M$ is trivial with fibre an infinite dimensional vector space V^∞. The bundle

$$\pi^\infty : J^\infty E_U = U \times V^\infty \to U \tag{12}$$

then has induced coordinates given by

$$(x^i, u^a, u^a_i, u^a_{i_1 i_2}, \ldots,). \tag{13}$$

We use multi-index notation and the summation convention throughout the paper. If $j^\infty\phi$ is the section of $J^\infty E$ induced by a section ϕ of the bundle E, then $u^a \circ j^\infty\phi = u^a \circ \phi$ and

$$u^a_I \circ j^\infty\phi = (\partial_{i_1}\partial_{i_2}...\partial_{i_r})(u^a \circ j^\infty\phi)$$

where r is the order of the symmetric multi-index $I = \{i_1, i_2, ..., i_r\}$,with the convention that, for $r = 0$, there are no derivatives.

The de Rham complex of differential forms $\Omega^*(J^\infty E, d)$ on $J^\infty E$ possesses a differential ideal, the ideal C of contact forms θ which satisfy $(j^\infty\phi)^*\theta = 0$

for all sections ϕ with compact support. This ideal is generated by the contact one-forms, which in local coordinates assume the form $\theta^a_J = du^a_J - u^a_{iJ}dx^i$. Contact one-forms of order 0 satisfy $(j^1\phi)^*(\theta) = 0$. In local coordinates, contact forms of order zero assume the form $\theta^a = du^a - u^a_i dx^i$.

Remarkably, using the contact forms, we see that the complex $\Omega^*(J^\infty E, d)$ splits as a bicomplex (though the finite level complexes $\Omega^*(J^p E)$ do not). The bigrading is described by writing a differential p-form α as an element of $\Omega^{r,s}(J^\infty E)$, with $p = r + s$ when $\alpha = \alpha^{\mathbf{J}}_{IA}(\theta^A_{\mathbf{J}} \wedge dx^I)$ where

$$dx^I = dx^{i_1} \wedge ... \wedge dx^{i_r} \quad \text{and} \quad \theta^A_{\mathbf{J}} = \theta^{a_1}_{J_1} \wedge ... \wedge \theta^{a_s}_{J_s}. \tag{14}$$

We intend to restrict the complex Ω^* by requiring that the functions $\alpha^{\mathbf{J}}_{IA}$ be local functions in the following sense.

Definition 8 *A* **local function** *on $J^\infty E$ is the pullback of a smooth function on some finite jet bundle $J^p E$, i.e. a composite $J^\infty E \to J^p E \to M$. In local coordinates, a local function $L(x, u^{(p)})$ is a smooth function in the coordinates x^i and the coordinates u^a_I, where the order $|I| = r$ of the multi-index I is less than or equal to some integer p.*

The **space of local functions** *will be denoted $Loc(E)$, while the subspace consisting of functions $(\pi^\infty_M)^* f$ for $f \in C^\infty M$ is denoted by Loc_M.*

Henceforth, the coefficients of all differential forms in the complex $\Omega^*(J^\infty E, d)$ are required to be local functions, i.e., for each such form α there exists a positive integer p such that α is the pullback of a form of $\Omega^*(J^p E, d)$ under the canonical projection of $J^\infty E$ onto $J^p E$. In this context, the horizontal differential is obtained by noting that $d\alpha$ is in $\Omega^{r+1,s} \oplus \Omega^{r,s+1}$ and then denoting the two pieces by, respectively, $d_H\alpha$ and $d_V\alpha$. One can then write

$$d_H\alpha = (-1)^s\{D_i\alpha^{\mathbf{J}}_{IA}\theta^A_{\mathbf{J}} \wedge dx^i \wedge dx^I + \alpha^{\mathbf{J}}_{IA}\theta^A_{\mathbf{J}i} \wedge dx^i \wedge dx^I\}, \tag{15}$$

where

$$\theta^A_{\mathbf{J}i} = \sum_{r=1}^{s}(\theta^{a_1}_{J_1} \wedge ...\theta^{a_r}_{J_r i} ... \wedge \theta^{a_s}_{J_s})$$

and where

$$D_i = \frac{\partial}{\partial x^i} + u^a_{iJ}\frac{\partial}{\partial u^a_J} \tag{16}$$

is the total differential operator acting on local functions.

We will work primarily with the d_H subcomplex, the algebra of horizontal forms $\Omega^{*,0}$, which is the exterior algebra in the dx^i with coefficients that are local functions. In this case we often use Olver's notation D for the horizontal differential $d_H = dx^i D_i$ where D_i is defined above. In addition to this notation, we also utilize the operation $\lrcorner$ which is defined as follows. Given any differential r-form α on a manifold N and a vector field X on N, $X \lrcorner\ \alpha$ denotes that (r-1)-form whose value at any $x \in N$ and $(v_1, ..., v_{r-1}) \in (T_xN \times \cdots \times T_xN)$ is $\alpha_x(X_x, v_1, ..., v_{r-1})$. We will sometimes use the notation $X(\alpha)$ in place of $X \lrcorner\ \alpha$.

Let ν denote a fixed volume form on M and let ν also denote its pullback $(\pi_E^\infty)^*(\nu)$ to $J^\infty E$ so that ν may be regarded either as a top form on M or as defining elements $P\nu$ of $\Omega^{n,0}E$ for each $P \in C^\infty(J^\infty E)$. We will almost invariably assume that $\nu = d^n x = dx^1 \wedge \cdots \wedge dx^n$.

It is useful to observe that for $R^i \in Loc(E)$ and

$$\alpha = (-1)^{i-1} R^i (\frac{\partial}{\partial x^i} \lrcorner\ d^n x), \tag{17}$$

then

$$d_H \alpha = (-1)^{i-1} D_j R^i [dx^j \wedge (\frac{\partial}{\partial x^i} \lrcorner\ d^n x)] = D_j R^j d^n x. \tag{18}$$

Hence, a total divergence $D_j R^j$ may be represented (up to the insertion of a volume $d^n x$) as the horizontal differential of an element of $\Omega^{n-1,0}(J^\infty E)$. It is easy to see that the converse is true so, that, in local coordinates, one has that total divergences are in one-to-one correspondence with D-exact n-forms.

Definition 9 *A* **local functional**

$$\mathcal{L}[\phi] = \int_M L(x, \phi^{(p)}(x)) dvol_M = \int_M (j^\infty \phi)^* L(x, u^{(p)}) dvol_M \tag{19}$$

is the integral over M of a local function evaluated for sections ϕ of E of compact support.

The **space of local functionals** *$\mathcal{F}$ is the vector space of equivalence classes of local functionals, where two local functionals are equivalent if they agree for all sections of compact support.*

If one does not want to restrict oneself to the case where the base space is a subset of $\mathbf{R}^n$, one has to take the transformation properties of the integrands under coordinate transformations into account and one has to integrate a horizontal n-form rather than a multiple of dx^n by an element of $Loc(E)$.

Lemma 10 *The vector space of local functionals $\mathcal{F}$ is isomorphic to the cohomology group $H^n(\Omega^{*,0}, D)$.*

Proof: Recall that one has a natural mapping $\hat{\eta}$ from $\Omega^{n,0}(J^\infty E)$ onto $\mathcal{F}$ defined by

$$\hat{\eta}(P\nu)(\phi) = \int_M (j^\infty\phi)^*(P)\nu \qquad \forall\phi \in \Gamma E. \tag{20}$$

It is well-known (see e.g. [18]) that $\hat{\eta}(P\nu)[\phi] = 0$ for all ϕ of compact support if and only if in coordinates P may be represented as a divergence, i.e., iff $P = D_i R^i$ for some set of local functions $\{R^i\}$. Hence, $\hat{\eta}(P\nu) = 0$ if and only if there exists a form $\beta \in \Omega^{n-1,0}$ such that the horizontal differential d_H maps β to $P\nu$. So the kernel of $\hat{\eta}$ is precisely $d_H\Omega^{n-1,0}$ and $\hat{\eta}$ induces an isomorphism from $H^n(d_H) = \Omega^{n,0}/d_H\Omega^{n-1,0}$ onto the space $\mathcal{F}$ of local functionals. $\square$

For later use, we also note that the kernel of $\hat{\eta}$ coincides with the kernel of the Euler-Lagrange operator: for $1 \leq a \leq m$, let E_a denote the a-th component of the Euler-Lagrange operator defined for $P \in Loc(E)$ by

$$E_a(P) = \frac{\partial P}{\partial u^a} - \partial_i \frac{\partial P}{\partial u^a_i} + \partial_i\partial_j \frac{\partial P}{\partial u^a_{ij}} - ... = (-D)_I(\frac{\partial P}{\partial u^a_I}). \tag{21}$$

The set of components $\{E_a(P)\}$ are in fact the components of a covector density with respect to the generating set $\{\theta^a\}$ for C_0, the ideal generated by the contact one-forms of order zero. Consequently, the Euler operator

$$E(Pd^n x) = E_a(P)(\theta^a \wedge d^n x), \tag{22}$$

for $\{\theta^a\}$ a basis of C_0, gives a well-defined element of $\Omega^{n,1}$. We have $E(P\nu) = 0$ iff $P\nu = d_H\beta$. For convenience we will also extend the operator E to map local functions to $\Omega^{n,1}$, so that $E(P)$ is defined to be $E(Pd^n x)$ for each $P \in Loc(E)$.

In section 2, we have assumed that we have a resolution of

$\mathcal{F} \simeq H^n(\Omega^{*,0}, d_H)$. In the case where M is contractible, such a resolution necessarily exists and is provided by the following (exact) extension of the horizontal complex $\Omega^{*,0}(Loc(E), d_H)$:

$$\begin{array}{ccccccccccc}
\mathbf{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{n-2,0} & \xrightarrow{d_H} & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0} \\
\downarrow\eta & & \downarrow\eta & & & & \downarrow\eta & & \downarrow\eta & & \downarrow\eta \\
0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H_0 = \mathcal{F}
\end{array}$$

Alternatively, we can achieve a resolution by taking out the constants: the space X_* is given by $X_i = \Omega^{n-i,0}$, for $0 \le i < n$, $X_n = \Omega^{0,0}/\mathbf{R}$, and $X_i = 0$, for $i > n+1$. Either way, we have a resolution of $\mathcal{F}$ and can proceed with the construction of an sh Lie structure. (For general vector bundles E, the assumption that such a resolution exists imposes topological restrictions on E which can be shown to depend only on topological properties of M [1].)

3.2 Poisson brackets for local functionals

To begin to apply the results of section 2, we must have a bilinear skew-symmetric mapping $\tilde{l}_2$ from $\Omega^{n,0} \times \Omega^{n,0}$ to $\Omega^{n,0}$ such that:

(i) $\tilde{l}_2(\alpha, d_H\beta)$ belongs to $d_H\Omega^{n-1,0}$ for all $\alpha \in \Omega^{n,0}$ and $\beta \in \Omega^{n,0}$, and

(ii) $\sum_{\sigma \in unsh(2,1)} (-1)^{|\sigma|} \tilde{l}_2(\tilde{l}_2(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}), \alpha_{\sigma(3)})$ belongs to $d_H\Omega^{n-1,0}$ for all $\alpha_1, \alpha_2, \alpha_3 \in \Omega^{n,0}$.

To introduce a candidate $\tilde{l}_2$, we define additional concepts. We say that X is a **generalized vector field over** M iff X is a mapping which factors through the differential of the projection of $J^\infty E$ to $J^r E$ for some non-negative integer r and which assigns to each $w \in J^\infty E$ a tangent vector to M at $\pi^\infty_M(w)$. Similarly Y is a **generalized vector field over** E iff Y also factors through $J^r E$ for some r and assigns to each $w \in J^\infty E$ a vector tangent to E at $\pi^\infty_E(w)$. In local coordinates one has

$$X = X^i \partial/\partial x^i \qquad Y = Y^j \partial/\partial x^j + Y^a \partial/\partial u^a \tag{23}$$

where $X^i, Y^j, Y^a \in Loc(E)$. A generalized vector field Q on E is called an evolutionary vector field iff $(d\pi)(Q_w) = 0$ for all $w \in J^\infty E$. In adapted coordinates, an evolutionary vector field assumes the form $Q = Q^a(w)\partial/\partial u^a$.

Given a generalized vector field X on M, there exists a unique vector field denoted Tot(X) such that $(d\pi^\infty_M)(Tot(X)) = X$ and $\theta(Tot(X)) = 0$ for every

contact one-form θ. In the special case that $X = X^i\partial/\partial x^i$, it is easy to show that $Tot(X) = X^i D_i$. We say that Z is a first order total differential operator iff there exists a generalized vector field X on M such that $Z = Tot(X)$. More generally, a total differential operator Z is by definition the sum of a finite number of finite order operators Z_α for which there exists functions $Z^J_\alpha \in Loc(E)$ and first order total differential operators $W_1, W_2, ...W_p$ on M such that

$$Z_\alpha = Z^J_\alpha(W_{j_1} \circ W_{j_2} \circ ... \circ Wj_p) \tag{24}$$

where $J = \{j_1, j_2, ..., j_p\}$ is a fixed set of multi-indices depending on α ($p = 0$ is allowed).

In particular, in adapted coordinates, a total differential operator assumes the form $Z = Z^J D_J$, where $Z^J \in Loc(E)$ for each multi-index J, and the sum over the multi-index J is restricted to a finite number of terms.

In an analogous manner, for every evolutionary vector field Q on E, there exists its **prolongation**, the unique vector field denoted $pr(Q)$ on $J^\infty E$ such that $(d\pi^\infty_E)(pr(Q)) = Q$ and $\mathcal{L}_{pr(Q)}(C) \subseteq C$, where $\mathcal{L}_{pr(Q)}$ denotes the Lie derivative operator with respect to the vector field $pr(Q)$ and C is the ideal of contact forms on $J^\infty E$. In local adapted coordinates, the prolongation of an evolutionary vector field $Q = Q^a\partial/\partial u^a$ assumes the form $pr(Q) = (D_J Q^a)\partial/\partial u^a_J$. The set of all total differential operators will be denoted by $TDO(E)$ and the set of all evolutionary vector fields by $Ev(E)$. Both $TDO(E)$ and $Ev(E)$ are left $Loc(E)$ modules.

One may define a new total differential operator Z^+ called the **adjoint** of Z by

$$\int_M (j^\infty\phi)^*[SZ(T)]\nu = \int_M (j^\infty\phi)^*[Z^+(S)T]\nu \tag{25}$$

for all sections $\phi \in \Gamma E$ and all $S, T \in Loc(E)$. It follows that

$$[SZ(T)]\nu = [Z^+(S)T]\nu + d_H\zeta \tag{26}$$

for some $\zeta \in \Omega^{n-1,0}(E)$. If $Z = Z^J D_J$ in local coordinates, then $Z^+(S) = (-D)_J(Z^J S)$. This follows from an integration by parts in (26) and the fact that (26) must hold for all T (see e.g. [18] corollary 5.52).

Assume that ω is a mapping from $C_0 \times C_0$ to $TDO(E)$ which is a module homomorphism in each variable separately. The adjoint of ω denoted ω^+ is

the mapping from $C_0 \times C_0$ to $TDO(E)$ defined by $\omega^+(\theta_1, \theta_2) = \omega(\theta_2, \theta_1)^+$. In particular ω is skew-adjoint iff $\omega^+ = -\omega$.

Using the module basis $\{\theta^a\}$ for C_0 , we define the total differential operators $\omega^{ab} = \omega(\theta^a, \theta^b)$. From these operators, we construct a bracket on the set of local functionals [9, 10, 11, 12, 13] (see e.g. [18, 5] for reviews) by

$$\{\mathcal{P}, \mathcal{Q}\} = \int_M \omega(\theta^a, \theta^b)(E_b(P))E_a(Q)d^n x, \tag{27}$$

where $\mathcal{P} = Pd^n x$ and $\mathcal{Q} = Qd^n x$ for local functions P and Q. As in other formulas of this type, it is understood that the local functional $\{\mathcal{P}, \mathcal{Q}\}$ is to be evaluated at a section ϕ of the bundle $E \to M$ and that the integrand is pulled back to M via $j^\infty\phi$ before being integrated.

We find it useful to introduce the condensed notation $\omega(E(P))E(Q) = \omega(\theta^a, \theta^b)(E_b(P))E_a(Q)$ throughout the remainder of the paper. In order to express $\omega(E(P))E(Q)$ in a coordinate invariant notation, note that $pr(\frac{\partial}{\partial u^a}) \lrcorner\ E(L) = E_a(L)d^n x$ for each local function$L$. Consequently, if $*$ is the operator from $\Omega^{n,0}E$ to $Loc(E)$ defined by $*(P\nu) = P$, then

$$\omega(E(P))E(Q) = \omega(\theta^a, \theta^b)(*[pr(\frac{\partial}{\partial u^b}) \lrcorner\ E(P)])(*[pr(\frac{\partial}{\partial u^a}) \lrcorner\ E(Q)]). \tag{28}$$

If coordinates on M are chosen such that $\nu = d^n x$, then it follows that

$$\{\mathcal{P}, \mathcal{Q}\} = \int_M \omega(\theta^a, \theta^b)(*[pr(Y_b) \lrcorner\ E(P)])(*[pr(Y_a) \lrcorner\ E(Q)])\nu, \tag{29}$$

where $\{Y_a\}$ and $\{\theta^b\}$ are required to be local bases of $Ev(E)$ and C_0, respectively, such that $\theta^b(Y_a) = \delta^b_a$. It is easy to show that the integral is independent of the choices of bases and consequently, one has a coordinate-invariant description of the bracket for local functionals.

3.3 Associated sh Lie algebra on the horizontal complex

This bracket for functionals provides us with some insight as to how $\tilde{l}_2$ may be defined; namely for $\alpha_1 = P_1\nu$ and $\alpha_2 = P_2\nu \in \Omega^{n,0}$, define $\tilde{l}_2(\alpha_1, \alpha_2)$ to be the skew-symmetrization of the integrand of $\{\mathcal{P}_1, \mathcal{P}_2\}$:

$$\tilde{l}_2(\alpha_1, \alpha_2) = \frac{1}{2}[\omega(E(P_1))E(P_2) - \omega(E(P_2))E(P_1)]\nu. \tag{30}$$

By construction, $\tilde{l}_2$ is skew-symmetric and, from $E(d_H\beta) = 0$ for $\beta \in \Omega^{n-1,0}$, it follows that $\tilde{l}_2(\alpha, d_H\beta) = 0$. Thus a strong form of property (i) required above for $\tilde{l}_2$ holds.

The symmetry properties of ω may be used to simplify the equation for $\tilde{l}_2(\alpha_1, \alpha_2)$. Skew-adjointness of ω implies

$$\omega(E(P_1))E(P_2)\nu = -\omega(E(P_2))E(P_1)\nu + d_H\gamma \tag{31}$$

for some $\gamma \in \Omega^{n-1,0}$, which depends on $\alpha_1 = P_1\nu$ and $\alpha_2 = P_2\nu$. In fact, since $E(d_H\beta) = 0$, the element γ depends only on the cohomology classes $\mathcal{P}_1, \mathcal{P}_2$ of α_1 and α_2. A specific formula for γ can be given by straightforward integrations by parts.

Hence, from (30) and (31), we get

$$\tilde{l}_2(\alpha_1, \alpha_2) = \omega(E(P_1))E(P_2)\nu - \frac{1}{2}d_H\gamma(\mathcal{P}_1, \mathcal{P}_2). \tag{32}$$

Furthermore, since $\int_M (j^\infty\phi)^* d_H\gamma = 0$ for all $\phi \in \Gamma E$, we see that

$$\{\mathcal{P}_1, \mathcal{P}_2\}(\phi) = \int_M (j^\infty\phi)^*[\omega(E(P_1))E(P_2)]\nu = \int_M (j^\infty\phi)^*\tilde{l}_2(\alpha_1, \alpha_2). \tag{33}$$

In order to explain the conditions necessary for $\tilde{l}_2$ to satisfy the required "Jacobi" condition, we formulate the problem in terms of "Hamiltonian" vector fields (see e.g. [18] chapter 7.1 or [5] chapter 2.5) and their corresponding Lie brackets.

Given a local function Q, one defines an evolutionary vector field $v_{\omega EQ}$ by

$$v_{\omega EQ} = \omega^{ab}(E_b(Q))\partial/\partial u^a = \omega(\theta^a, \theta^b)(*[pr(\partial/\partial u^b)\lrcorner\ E(Q)])\partial/\partial u^a. \tag{34}$$

Again, the vector field $v_{\omega EQ}$ depends only on the functional $\mathcal{Q}$ and not on which representative Q one chooses in the cohomology class $\mathcal{Q} \sim Q\nu + d_H\Omega^{n-1,0}$. Thus, for a given functional $\mathcal{Q}$, let $\hat{v}_{\mathcal{Q}} = v_{\omega EQ}$ for any representative Q.

Since $\{\mathcal{P}_1, \mathcal{P}_2\} = \int_M \tilde{l}_2(\alpha_1, \alpha_2)$, we see that

$$\hat{v}_{\{\mathcal{P}_1, \mathcal{P}_2\}} = v_{\omega E(\tilde{l}_2(\alpha_1, \alpha_2))}. \tag{35}$$

Note also that

$$\omega(E(P_1))E(P_2) = \omega^{ab}(E_b(P_1))E_a(P_2) = pr[\omega^{ab}E_b(P_1)\partial/\partial u^a] \lrcorner\ E(P_2)$$
$$= pr(v_{\omega E(P_1)}) \lrcorner\ E(P_2) = pr(\hat{v}_{\mathcal{P}_1}) \lrcorner\ E(P_2). \quad (36)$$

Moreover, integration by parts allows us to show that

$$pr(Q)(P)\nu = pr(Q) \lrcorner\ d(P\nu) = pr(Q) \lrcorner\ E(P\nu) + d_H(pr(Q) \lrcorner\ \sigma), \quad (37)$$

for arbitrary evolutionary vector fields Q and local functions P, and for some form $\sigma \in \Omega^{n-1,0}$ depending on P. For every such Q, let I_Q denote a mapping from $\Omega^{n,0}$ to $\Omega^{n-1,0}$ such that

$$pr(Q)(P)\nu = pr(Q) \lrcorner\ E(P) + d_H(I_Q(P\nu)) \quad (38)$$

for all $P\nu \in \Omega^{n,0}$. Explicit coordinate expressions for I_Q can be found in [18] chapter 5.4 or in [5] chapter 17.5 .

It follows from the identities (32), (36) and (38) that

$$\tilde{l}_2(\alpha_1, \alpha_2) = pr(\hat{v}_{\mathcal{P}_1})(P_2)\nu - d_H I_{\hat{v}_{\mathcal{P}_1}}(P_2\nu) - \frac{1}{2}d_H\gamma(\mathcal{P}_1, \mathcal{P}_2)). \quad (39)$$

Thus, for $\alpha_1, \alpha_2, \alpha_3 \in \Omega^{n,0}$, we see that

$$\tilde{l}_2(\tilde{l}_2(\alpha_1, \alpha_2), \alpha_3) = -\tilde{l}_2(\alpha_3, \tilde{l}_2(\alpha_1, \alpha_2)) =$$
$$= -\tilde{l}_2(\alpha_3, pr(\hat{v}_{\mathcal{P}_1})(P_2)\nu - d_H(\cdot)) = -\tilde{l}_2(\alpha_3, pr(\hat{v}_{\mathcal{P}_1})(P_2)\nu) \quad (40)$$

and

$$\tilde{l}_2(\tilde{l}_2(\alpha_1, \alpha_2), \alpha_3) = -pr(\hat{v}_{\mathcal{P}_3})(pr(\hat{v}_{\mathcal{P}_1})(P_2))\nu + d_H\zeta, \quad (41)$$

where ζ is given by

$$\zeta(\mathcal{P}_1, P_2, \mathcal{P}_3) = I_{\hat{v}_{\mathcal{P}_3}}(pr(\hat{v}_{\mathcal{P}_1})(P_2)\nu) + \frac{1}{2}\gamma(\mathcal{P}_3, \{\mathcal{P}_1, \mathcal{P}_2\}). \quad (42)$$

Rewriting the left hand side of the Jacobi identity in Leibnitz form and using (35), (39) and (41), we find

$$\sum_{\sigma \in unsh(2,1)} (-1)^{|\sigma|}\tilde{l}_2(\tilde{l}_2(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}), \alpha_{\sigma(3)}) =$$
$$= -\tilde{l}_2(\alpha_3, \tilde{l}_2(\alpha_1, \alpha_2)) - \tilde{l}_2(\tilde{l}_2(\alpha_1, \alpha_3), \alpha_2) + \tilde{l}_2(\alpha_1, \tilde{l}_2(\alpha_3, \alpha_2)) =$$
$$= [-pr(\hat{v}_{\mathcal{P}_3})(pr(\hat{v}_{\mathcal{P}_1})(P_2)) + pr(\hat{v}_{\mathcal{P}_1})(pr(\hat{v}_{\mathcal{P}_3})(P_1))$$
$$-pr(\hat{v}_{\{\mathcal{P}_1, \mathcal{P}_3\}})(P_2)]\nu + d_H\eta, \quad (43)$$

with

$$\eta(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) = \zeta(\mathcal{P}_1, P_2, \mathcal{P}_3) - \zeta(\mathcal{P}_3, P_2, \mathcal{P}_1) \\ + I_{\hat{v}_{\{\mathcal{P}_1,\mathcal{P}_3\}}}(P_2\nu) + \frac{1}{2} d_H \gamma(\{\mathcal{P}_1, \mathcal{P}_3\}, \mathcal{P}_2). \tag{44}$$

Although ζ depends on the representative P_2 and not its cohomology class, η depends only on the cohomology classes $\mathcal{P}_i$ because it is completely skew-symmetric.

It follows from this identity that if

$$pr(\hat{v}_{\{\mathcal{P}_1,\mathcal{P}_2\}}) = [pr(\hat{v}_{\mathcal{P}_1}), pr(\hat{v}_{\mathcal{P}_2})] \tag{45}$$

for all $\mathcal{P}_1, \mathcal{P}_2$, then $\{\cdot, \cdot\}$ satisfies the Jacobi condition. Under these conditions, the mapping $H : \mathcal{F} \longrightarrow Ev(E)$ defined by $H(\mathcal{P}) = \hat{v}_{\mathcal{P}}$ is said to be **Hamiltonian**. Equivalent conditions on the mapping H alone for the bracket $\{\cdot, \cdot\}$ to be a Lie bracket can be found in [18, 5]. The derivation given here allows us to give, in local coordinates, an explicit form for the exact term (44) violating the Jacobi identity.

If H is Hamiltonian, the bracket $\tilde{l}_2$ satisfies condition (ii) and the construction of section 2 applies. Because the resolution stops with the horizontal zero-forms, we get a possibly non-trivial $L(n+2)$ algebra on the horizontal complex. If we remove the constants, we can then extend to a full $L_\infty-$algebra by defining the further l_i to be 0. In addition, because property (i) holds without any l_1 exact term on the right hand side, remark (ii) at the end of section 2.1 applies, i.e., we need only two terms in the defining equations of the sh Lie algebra and the maps l_k induce well-defined higher order maps on the space of local functionals. On the other hand, if we do not remove the constants, the operation l_{n+2} takes values in $X_n = \Omega^{0,0} = Loc(E)$ and induces a multi-bracket on $H^n(\Omega^{*,0}, d_H) \simeq \mathcal{F}$, the space of local functionals, with values in $H_n(X_*, l_1) = H^0(\Omega^{*,0}, d_H) \simeq H_{DR}(C^\infty(M)) = \mathbf{R}$.

We have thus proved the following main theorem.

Theorem 11 *Suppose that the horizontal complex without the constants* $(\Omega^{*,0}/\mathbf{R}, d_H)$ *is a resolution of the space of local functionals* $\mathcal{F}$ *equipped with a Poisson bracket as above. If the mapping* H *from* $\mathcal{F}$ *to evolutionary vector fields is Hamiltonian, then to the Lie algebra* $\mathcal{F}$ *equipped with the induced*

bracket $\hat{l}_2 = \{\cdot, \cdot\}$, *there correspond maps* $l_i : (\Omega^{*,0}/\mathbf{R})^{\otimes i} \to \Omega^{*-i+2,0}/\mathbf{R}$ *for* $1 \leq i \leq n+2$ *satisfying the sh Lie identities*

$$l_1 l_k + l_{k-1} l_2 = 0.$$

The corresponding map $\hat{l}_{n+2}$ *on* $\mathcal{F}^{\otimes n+2}$ *with values in* $H^0(\Omega^{*,0}, d_H) = \mathbf{R}$ *satisfies*

$$\hat{l}_{n+2} \hat{l}_2 = 0.$$

Specific examples for $n = 1$ are worked out in careful detail by Dickey [6].

4 Conclusion

The approach of Gel'fand, Dickey and Dorfman to functionals and Poisson brackets in field theory has the advantage of being completely algebraic. In this paper, we have kept explicitly the boundary terms violating the Jacobi identity for the bracket of functions, instead of throwing them away by going over to functionals at the end of the computation. In this way, we can work consistently with functions alone, at the price of deforming the Lie algebra into an sh Lie algebra. Our hope is that this approach will be useful for a completely algebraic study of deformations of Poisson brackets in field theory.

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